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# I Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17 Scheme)
## MATHEMATICS
## M101T : Algebra – I

Time : 3 Hours Max. Marks : 70

***Instructions :*** *1) Answer **any five full** questions from the following.*
*2) **All** questions carry **equal** marks.*

1. a) Define a permutation group. Show that every permutation on a finite set is a product of disjoint cycles.

   b) Let $\phi : G \to G'$ be a homomorphism with Kernel K and let $\bar{N}$ be a normal subgroup of $\bar{G}$ and $N = \{g \in G : \phi(g) \in \bar{N}\}$. Prove that $G/N \cong \bar{G}/\bar{N}$.

   c) State and prove the Cayley's theorem for permutation group. **(4+5+5)**

2. a) State and prove the Cauchy-Frobenius Lemma.

   b) Derive the class equation of finite group G. Verify the class equation of symmetric group $S_3$.

   c) Prove that every group of order $p^2$, for a prime p, is abelian. **(4+6+4)**

3. a) Define a p-sylow subgroup of a group. If p is a prime number and $p^{\alpha} | o(G)$, then show that G has a subgroup of order $p^{\alpha}$.

   b) Let o(G) = pq, where p and q are distinct primes with p < q and $q \neq 1 \pmod{p}$, then prove that G is cyclic and hence abelian. **(8+6)**

4. a) Define a simple group. Show that a normal subgroup N of G is maximal if and only if the quotient group G/N is simple. Further, show that the symmetric group $S_3$ is not simple.

   b) State and prove the Jordan-Holder Theorem. **(6+8)**

P.T.O.

5. a) Let R be a commutative ring with unity whose ideals are {0} and R itself. Then show that R is a field.

   b) Let U be a left ideal of a ring R and $\lambda(U) = \{x \in R : xu=0 \text{ for all } u \in U\}$. Then show that $\lambda(U)$ is an ideal of R.

   c) Let R and $R'$ be rings and $\phi$ is a homomorphism of R onto $R'$ with kernel U. Then prove that $R' \approx R/U$. Further, show that there is one to one correspondence between the set of ideals $W'$ of $R'$ and the set of ideals W of R containing U. **(4+4+6)**

6. a) Show that a ring Z of integers is a principal ideal ring.

   b) If R is a commutative ring with unity and M is an ideal of R, then show that M is a maximal ideal of R if and only if R/M is a field.

   c) Prove that any two isomorphic integral domains have isomorphic quotient fields. **(4+5+5)**

7. a) Define an Euclidean ring. Prove that

      i) every field is an Euclidean ring.

      ii) the ring Z[i] of Gaussian integers is an Euclidean ring.

   b) State and prove the Unique factorization theorem.

   c) If p is a prime number of the form 4n +1, then show that $x^2 \equiv -1 \pmod{p}$. **(6+4+4)**

8. a) Define a primitive polynomial. Prove that the product of primitive polynomials is a primitive polynomial.

   b) State and prove the Eisenstein criterion for the irreducibility of a polynomial. **(7+7)**

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# I Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17)
## MATHEMATICS
## M102T : Real Analysis

Time : 3 Hours Max. Marks : 70

***Instructions :*** *1) Answer **any five full** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Show that $(3x + 1)$ is Riemann integrable on $[1, 2]$.

b) Prove that $f \in R[\alpha]$ on $[a, b]$ if and only if given $\in > 0$, there exists a partition $p$ of $[a, b]$ such that $U(p, f, \alpha) - L(p, f, \alpha) < \in$.

c) If $f \in R[\alpha_1]$ on $[a, b]$ and $f \in R[\alpha_2]$ on $[a, b]$, then prove that $f \in R[\alpha_1 + \alpha_2]$ on $[a, b]$. **(3+6+5)**

2. a) If $f_1, f_2 \in R[\alpha]$ on $[a, b]$ and $f_1 \le f_2$, then show that $\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$.

b) If f is continuous on $[a, b]$ and $\alpha$ is monotonically increasing function on $[a, b]$, then show that $f \in R[\alpha]$.

c) Let f be Riemann integrable on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt$, where $a \le x \le b$.

Then prove that F is continuous on $[a, b]$. Further, show that if $f(t)$ is continuous at a point $x_0$ on $[a, b]$. Then show that F is differentiable at $x_0$ and $F'(x_0) = f(x_0)$. **(4+4+6)**

3. a) Consider the functions $\beta_1(x)$ and $\beta_2(x)$ defined as follows :

$$\beta_1(x) = \begin{cases} 0, & \text{when } x \le 0 \\ 1, & \text{when } x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, & \text{when } x < 0 \\ 1, & \text{when } x \ge 0 \end{cases}$$

Verify whether $\beta_1(x) \in R[\beta_2(x)]$ on $[-1, 1]$.

b) If $\lim_{\mu(p) \to 0} S(p, f, \alpha)$ exists, then show that $f \in R[\alpha]$ on $[a, b]$ and $\lim_{\mu(p) \to 0} S(p, f, \alpha) = \int_a^b f d\alpha$.

c) Calculate the total variation functions of $f(x) = x - [x]$ on $[0, 2]$ where $[x]$ is the maximum integral function. **(7+3+4)**



P.T.O.

4. a) State and prove Weierstrass M-test.

b) Test for uniform convergence for $\left\{\frac{nx}{1+n^2x^2}\right\}$ on [0, 1].

c) Suppose $f_n \to f$ uniformly on [a, b] and if $x_0 \in [a, b]$ such that $\lim_{x\to x_0} f_n(x) = A_n$, n = 1, 2, 3, . . . then prove that

i) An converges.

ii) $\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x)$. **(5+4+5)**

5. a) If $\{f_n(x)\}$ is uniformly convergent to f(x) on [a, b] and each $f_n(x)$ is continuous on [a, b] then prove that f(x) is continuous on [a, b].

b) Let $\{f_n(x)\}$ be a sequence of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R$ [a, b]. Prove the following

i) $f(x) \in R$ [a, b]

ii) $\int_a^x \lim_{n\to\infty} f_n(t)dt = \lim_{n\to\infty} \int_a^x f_n(t)dt$. **(7+7)**

6. a) Define a K-cell in $\mathbb{R}^K$. Let $I_1 \supset I_2 \supset I_3 \supset \ldots$ be a sequence of K-cells in $\mathbb{R}^K$. Show that $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

b) Prove that every bounded infinite subset of $\mathbb{R}^K$ has a limit point in $\mathbb{R}^K$.

c) Let f(x) be a continuous real (or complex) valued function defined on [a, b]. Then show that there exists polynomials $\{P_n(x)\}_{n=1}^{\infty}$, such that $f(x) = \lim_{n\to\infty} P_n(x)$ uniformly on [a, b]. If f(x) is real valued then $P_n(x)$ are real valued polynomials. **(4+3+7)**

7. a) Let $E \subset R^n$ be an open set and $f : E \to R^m$ be a map. Prove that f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \le i \le m$, $1 \le j \le n$.

b) If $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, then prove that $\|T\| < \infty$ and T is a uniformly continuous mapping of $\mathbb{R}^n$ onto $\mathbb{R}^m$.

c) Let $f : [a, b] \to \mathbb{R}^k$, $f = (f_1, f_2, \ldots f_k)$, f is differentiable if and only if each $f_i$ is differentiable. **(6+5+3)**

8. State and prove the implicit function theorem. **14**

# First Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17)
## MATHEMATICS
## M103T : Topology – I

Time : 3 Hours Max. Marks : 70

***Instructions :*** *1) Answer* ***any five full*** *questions.*
*2)* ***All*** *questions carry* ***equal*** *marks.*

1. a) Show that every infinite set contains countable subset.

b) Show that union of denumerable collection of denumerable sets is denumerable.

c) With usual notations prove that

i) C . C = C

ii) $\wp_0 \cdot \wp_0 = \wp_0$. **[5+5+4]**

2. a) State and prove Cantor's theorem.

b) With usual notation prove that $2^{\wp_0} = C$.

c) Define :

i) Continuum hypothesis and

ii) Zorn's lemma. **[6+6+2]**

3. a) Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then show that d is metric on $\mathbb{R}^2$.

b) Describe the open and closed spheres for the discrete metric space. **[7+7]**

4. a) State and prove Cantor's Intersection theorem.

b) Show that every metric space has a completion. **[7+7]**

5. a) Let $(X, \tau)$ be a topological space and let A and B be two subsets of X. Then prove

i) $A \subseteq B \Rightarrow A^{\circ} \subseteq B^{\circ}$

ii) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

iii) $(A \cup B)^{\circ} \neq A^{\circ} \cup B^{\circ}$

b) Define a closed set. Show that an arbitrary intersection of closed sets is closed and a finite union of closed sets is closed. **[9+5]**



P.T.O.

6. a) Let X and Y be topological spaces. Let $f : X \to Y$ be a function then show that followings are equivalent :

   i) f is continuous.

   ii) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

   iii) For any closed set B of Y, the set $f^{-1}(B)$ is closed in X.

   b) State and prove pasting lemma. **[9+5]**

7. a) Let $(X, \tau_1)$ and $(Y, \tau_2)$ are two topological spaces defined as under

   $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d\}$

   $\tau_1 = \{X, \phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$

   $\tau_2 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$

   $f : X \to Y$ defined as $f(1) = b$, $f(2) = c$, $f(3) = d$, $f(4) = c$. Find whether f is continuous or not.

   b) Show that union of any family of connected sets having a non-empty intersection is a connected set. **[7+7]**

8. a) State and prove intermediate value theorem.

   b) Show that continuous image of a path connected space is path connected.

   c) Show that a path connected space X is connected. **[6+4+4]**

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# First Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17 Scheme)
## MATHEMATICS
## M104T : Ordinary Differential Equations

Time : 3 Hours Max. Marks : 70

***Instructions :*** *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let $\{\varphi_j(x)\}_{j=1,2,\ldots,n}$ be a fundamental set of $L_n y = 0$ on I. Then prove that $\{\psi_j(x) : j = 1, 2, \ldots, n\}$ is a fundamental set of $L_n y = 0$ on I if and only if there exists a non-singular constant matrix C of order n such that $\psi = c\,\varphi$, where $\psi = \{\psi_1(x), \ldots, \psi_n(x)\}$ and $\varphi = \{\varphi_1(x), \ldots, \varphi_n(x)\}$ : Further show that $W\{\psi_j(x) : j = 1, 2, \ldots, n\} = |C|\, W\{\varphi_j(x) : j = 1, 2, \ldots, n\}$.

   b) Find the Wronskian of the solutions of $2x^2y'' + 7xy' + 3y = 0$.

   c) Verify Liouville's formula for $x^2y'' - 7xy' + 15y = 0$. **(6+4+4)**

2. a) Verify Lagrange's identity for $x^2y'' - 3xy' + 3y = 0$.

   b) Find the general solution of $x^2y'' + 9xy' + 12y = 0$ by finding the solution of its adjoint equation.

   c) Using the method of variation of Parameters, solve $y'' + 4y = \operatorname{cosec} 2x$. **(5+5+4)**

3. a) Show that the differential equation $y'' + \frac{k}{x^2}y = 0; 0 \le x < \infty$, where k is a constant and $x > 0$ is oscillatory or non-oscillatory according as $k > 1/4$ or $k \le 1/4$.

   b) Show that the IVP $\frac{dy}{dx} = \begin{cases} \frac{4x^3y}{x^4+y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ with $y(0) = 0$ has infinitely many solutions.

   c) Obtain Green's function for the eigen value problem $y'' + \frac{1}{4}y = \sin 2x$; $y(0) = 0$, $y(\pi) = 0$. **(4+4+6)**



P.T.O.

4. a) Find the eigen values and eigen functions of $\frac{d}{dx}\left\{(x^2+1)\frac{dy}{dx}\right\}+\frac{\lambda}{x^2+1}y=0;$ $y(0)=0,\ y(1)=0$.

b) Show that the eigen values of a sub adjoint eigen value problem are real.

c) Find the eigen values and eigen functions of $y''+\lambda y=0;\ y(0)=0,\ y(\pi)=0,$ $(\lambda>0)$. Further expand x in terms of orthonormal eigen functions of the eigen value problem. **(5+4+5)**

5. a) Find the ordinary, regular and irregular singular points if any of the differential equation $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0,$ (n is a constant).

b) Find the power series solution of the IVP $(x^2-1)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+xy=0;\ y(0)=4,\ y'(0)=6$. **(7+7)**

6. a) Find the solution of the problem $2x^2y''+xy'+(x^2-3)y=0$ about a regular singular point.

b) Prove that the Hermite polynomials are orthogonal over the interval $(-\infty,\infty)$ with respect to the weight function $e^{-x^2}$. **(7+7)**

7. a) Find the fundamental matrix and also general solution of the matrix equation $\underset{\sim}{x}'(t)=A\,\underset{\sim}{x}(t)$, where $A=\begin{pmatrix}-1 & 2 & 3\\ 0 & -2 & 1\\ 0 & 3 & 0\end{pmatrix}$.

b) Obtain the solution $\psi(t)$ of the IVP $\underset{\sim}{x}'(t)=A\,\underset{\sim}{x}(t)+B(t),\ \psi(0)=\begin{pmatrix}0\\1\end{pmatrix}$, $A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix},\ B(t)=\begin{pmatrix}\text{Sint}\\ \text{Cost}\end{pmatrix}$. **(7+7)**

8. a) Locate the critical point and find the nature of the system $\frac{dx}{dt}=x+y,\ \frac{dy}{dt}=3x-y$. Also find the equation of phase path.

b) Determine the nature and stability of the critical points of $\frac{dx}{dt}=x+2y+x\cos y,\ \frac{dy}{dt}=-y-\sin y$.

c) Determine the stability of the critical point (0, 0) by using Liapunov direct method for $\frac{dx}{dt}=-x+y^2,\ \frac{dy}{dt}=-y+x^2$. **(5+5+4)**

# I Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17 Scheme)
## MATHEMATICS
## M105T : Discrete Mathematics

Time : 3 Hours Max. Marks : 70

***Instructions :*** *i) Answer* ***any five full*** *questions.*
*ii)* ***All*** *questions carry* ***equal*** *marks.*

1. a) Explain the contradiction method of proof, use it to prove that $\sqrt{5}$ is irrational.

b) Test the validity of the following arguments.

i)
$$\begin{array}{l} p \Leftrightarrow q \\ q \Leftrightarrow r \\ r \Leftrightarrow \sim s \\ \underline{\sim s \Rightarrow q} \\ \therefore s \end{array}$$

ii)
$$\begin{array}{l} (\sim p \vee q) \Rightarrow r \\ r \Rightarrow (s \vee t) \\ \sim s \wedge \sim u \\ \underline{\sim u \Rightarrow \sim t} \\ \therefore p \end{array}$$

c) Show that if 11 numbers are choosen from the set {1, 2, 3, ..., 20}, one of them is a multiple of another. **(4+6+4)**

2. a) Find the number of ways of forming a committee of 9 students drawn from 3 different classes. So that the students from the same class do not have the absolute majority in the committee.

b) How many ways are there to form a three letter sequence using the letters a, b, c, d, e, f

i) with repetition of letters allowed

ii) without repetition of any letter

iii) without repetition that contain the letter 'e'

iv) with repetition that contain the letter 'e' ?

c) Find the least number of ways of choosing 3 different number 1 to 10, so that all choices have the same sum. **(5+6+3)**

3. a) Solve the Recurrence relation $a_{n+2} - 4a_{n+1} + 4a_n = (n + 1)^2$ with initial condition $a_0 = 0$, $a_1 = 1$.

b) Write a short note on modelling "The tower of Hanoi" problem and find an explicit formula for the solution of the problem.

c) Determine the co-efficient of $x^5$ in $(1 - 2x)^{-7}$. **(5+4+5)**

4. a) Let A = {1, 2, 3, 4} and R be the binary relation on set A defined by R = {(a, b)/a $\leq$ b}. Write down 'R' as a set of ordered pairs, also write down the relation matrix and digraph corresponding to R.

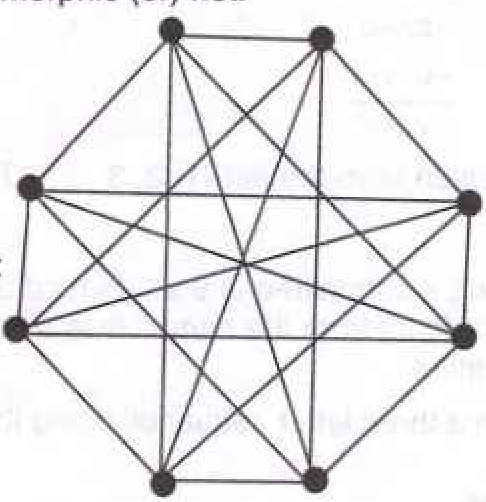
b) Define transitive closure of a relation. Find the transitive closure of the relation R = {(a, a), (a, b), (a, c), (c, d), (d, e), (b, d), (b, c), (e, b), (e, a)} defined on A = {a, b, c, d, e} using Warshall's algorithm.

c) Write the Hasse diagram for the set $D_{100}$ which consists of all divisors of 100 and R = {(x, y)/x, y $\in D_{100}$ and x divides y}. **(4+5+5)**

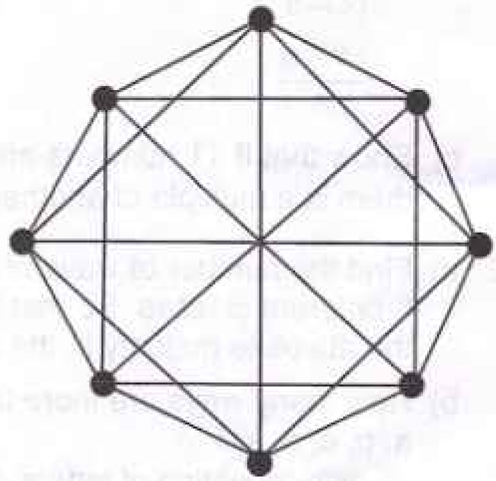
5. a) With usual notation, prove that $\delta(G) \leq \frac{2q}{p} \leq \Delta(G)$.

b) Define graph isomorphism. Check whether the following graphs are isomorphic (or) not.

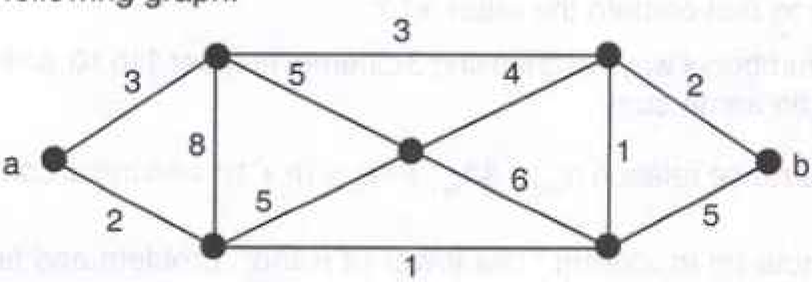
$G_1$:



$G_2$:



c) Find the shortest path between 'a' and 'b' using Dijkstra's algorithm for the following graph.

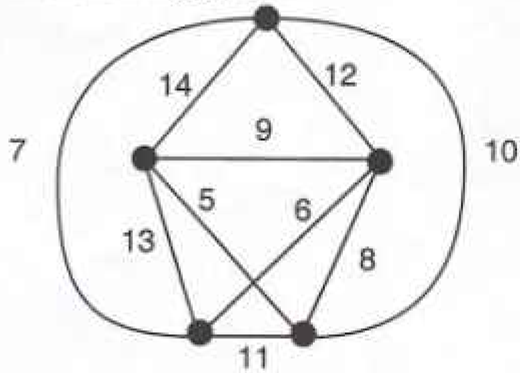



**(4+5+5)**

6. a) State and prove Ore's theorem for Hamiltonian graph.

b) Let G be a connected graph, then show that G contains an Eulerian trial if and only if G has exactly two odd vertices.

c) Using the nearest neighbour method, find the weight of a spanning cycle for the following graph.




(6+4+4)

7. a) If G is a connected plane graph with p – vertices, q – edges and r –faces, then prove that $p - q + r = 2$.

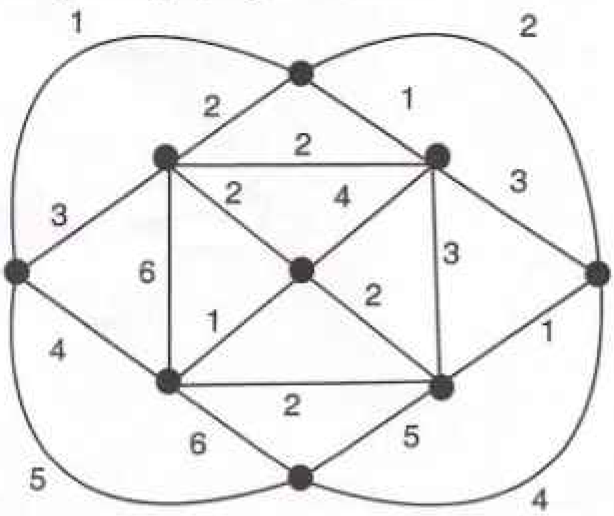
b) If G is a planar connected (p, q) graph without triangles, then prove that $q \leq 2p - 4$ for all $p \geq 3$.

c) Define vertex connectivity and edge connectivity of a graph with an example. (5+5+4)

8. a) Define binary tree. If a binary tree has p vertices of which k are pendant then prove that it has $p - k - 1$ vertices of degree 3.

b) Prove that a graph is connected if and only if it contains a spanning tree.

c) Write and apply Prim's algorithm to find a minimum spanning tree for the weighted graph given below.




(5+4+5)

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# I Semester M.Sc. Degree Examination, May 2024
## (CBCS – Y2K17)
## MATHEMATICS
## M107SC : Mathematical Analysis

Time : 3 Hours Max. Marks : 70

***Instructions** : i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Show that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(c)$ is closed in X for every closed set c in Y.

 b) If f is continuous on [a, b], $f'(c)$ exists at some point $c \in [a, b]$, g is defined on an interval which contains the range of f and g is differentiable at f(c), then show that $h(x) = g(f(x))$, $a \leq x \leq b$ is differentiable at c. Further, prove that $h'(c) = g'(f(c))f'(c)$.

 c) Prove that, there is no value k such that the equation $x^3 - 3x + k = 0$ has two distinct roots in [0, 1]. **(5+6+3)**

2. a) Show that continuous function on a compact metric space is uniformly continuous.

 b) Let E be a non compact set in R. Then show that

 i) There exists a continuous function on E, which is not bounded.

 ii) There exists a continuous and bounded function on E which has no maximum.

 c) Verify Lagrange's mean value theorem for the function $f(x) = \frac{x}{x-1}$ in (0, 2). **(5+5+4)**

P.T.O.

3. a) Let the function f be defined on (a, b). If f is discontinuous at a point x, then discuss about the cases of discontinuities with example.

   b) Plot the following functions

   i) $f(x)=\begin{cases} x+5; & -3\le x<-1 \\ -x+10; & -1\le x\le 1 \\ 2x+3; & 1\le x\le 3 \end{cases}$

   ii) $g(x)=\begin{cases} \sin x, & x\neq 0 \\ 1, & x=1 \end{cases}$

   Classify the discontinuities of the functions into first and second kinds.

   c) State and prove fixed point theorem. **(6+4+4)**

4. a) Discuss the following with an example

   i) Limit point

   ii) Interior point

   iii) Exterior point.

   b) Show that the function defined by

   $$f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

   is continuous at x = 0 but not differentiable at x = 0.

   c) State and prove intermediate value theorem for derivatives. **(6+4+4)**

5. a) Let $\{x_n\}$ be a monotonic sequence. Then show that $\{x_n\}$ is convergent if and only if it is bounded.

   b) Prove the following :

   i) If p > 1, then $\lim_{n\to\infty}\sqrt[n]{p}=1$.

   ii) If p > 0 and $\alpha$ is real then $\lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n}=0$.

   iii) If |x| < 1, then $\lim_{n\to\infty}x^n=0$.

   c) Show that the sequence

   $b_n=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}$ converges to zero. **(5+6+3)**

6. a) State and prove Cauchy's second theorem on limits.

   b) Show that the sequence $\left\{a_n^{1/n}\right\}$, where $a_n = \frac{(3n)!}{(n!)^3}$ converges and find its limit.

   c) Show that, if a series $\Sigma u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$. **(7+3+4)**

7. a) Let $1 + x + x^2 + \ldots$ be the geometric series. Then show that this series.

      i) Converges if $|x| < 1$.

      ii) Oscillates infinitely if $x < -1$.

   b) If a series $\Sigma u_n$ of positive monotonic decreasing terms converges and $u_n \to 0$ as $n \to 0$, then show that $\lim_{n\to\infty} nu_n = 0$.

   c) Test the convergence of the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$ **(6+5+3)**

8. a) Let $\Sigma u_n$ and $\Sigma v_n$ be two positive term series and there exists a positive integer 'm' such that

   $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}\ \forall n \geq m$

   Then prove the following

      i) $\Sigma u_n$ is convergent, if $\Sigma v_n$ is convergent

      ii) $\Sigma v_n$ is divergent, if $\Sigma u_n$ is divergent.

   b) State Cauchy's root test for series and prove that

   $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is convergent.

   c) Test the convergence of the series $\sum \frac{1}{n^2+1}$. **(6+5+3)**

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