PG – 447

III Semester M.Sc. Degree Examination, May 2024
(CBCS – Y2K17 Scheme)
MATHEMATICS
M305T : Numerical Analysis – II

Time : 3 Hours　　　　　　　　　　Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Using Picard's method, solve the initial value problem $\frac{d^2y}{dx^2} + xy = 0$; $y(0) = 1$, $\frac{dy(0)}{dx} = 0$ and hence estimate $y(0.05)$.

   b) Solve $\frac{dy}{dx} = y + e^x$ with $y(0) = 0$, find $y(0.4)$ taking $h = 0.2$ by using modified Euler's method. **(7+7)**

2. a) Solve $y'' = y + x$ with $y(0) = 0$, $y'(0) = 0$ by Runge-kutta method of two slopes and 4 slopes.

   b) Discuss the relative and absolute stability of Runge-kutta method of 2nd and 4th order. **(7+7)**

3. a) Derive the three-step Adam-Bashforth method for $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$.

   b) Solve $\frac{d^2y}{dx^2} = x + y$; $y(0) = 0$, $y(1) = 1$ by using finite difference method, take $\Delta x = 0.25$ (D0 3 iterations by Gauss-Seidel method). **(7+7)**

4. Describe the Shooting method for the solution of higher order differential equation. Hence solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ subject to $y(0) = 0$, $y(0.2) = 1$. **14**

5. a) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $0 \le x \le 1$, $t \ge 0$ subject to $u(x, 0) = \sin\pi x$, $0 \le x \le 1$ and $u(0, t) = 0 = u(1, t)$ by Schmidt method choosing $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{4}$. Obtain the solution at 2nd time level.

P.T.O.

b) Solve $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ with $0 \le x \le 1, t \ge 0$ subject to $u(x, 0) = x(1 - x);\ 0 \le x \le 1$ and $u(0, t) = 0 = u(1, t);\ t \ge 0$ by Dufort-Frankel method taking $\Delta x = \frac{1}{4},\ \Delta t = \frac{1}{64}$. Obtain solution at first time level. **(7+7)**

6. a) Solve the boundary value problem $u_{xx} + u_{yy} = \sin(\pi x)\sin(\pi y);\ 0 \le x \le 1,$ $0 \le y \le 1$, subject to the condition u = 0 on the boundary take $\Delta x = \Delta y = \frac{1}{3}$ and hence solve the system of equations by Gauss-Seidel method.

   b) Show that the Crank-Nicolson method is unconditionally stable. **(7+7)**

7. a) Solve $u_{tt} = 4u_{xx};\ 0 \le x \le 1$ and $t \ge 0$ subject to $u(x, 0) = \sin\pi x,\ u_t(x, 0) = 0;\ 0 \le x \le 1$ and $u(0, t) = 0 = u(1, t);\ t \ge 0$. Choose $\Delta x = \frac{1}{4},\ \Delta t = \frac{1}{64}$. Obtain solution at 2[nd] time level.

   b) Discuss the stability of the finite difference method applied to wave equation. **(7+7)**

8. Solve the 2D partial differential equation $u_t = u_{xx} + u_{yy};\ 0 \le x, y \le 1$ subject to the condition $u(x, y, 0) = \sin\pi x \sin\pi y$ and u = 0 on the boundary points using ADI method given $\Delta x = \Delta y = \frac{1}{3}, \Delta t = \frac{1}{72}$. **14**

PG – 446

III Semester M.Sc. Degree Examination, May 2024
(CBCS – Y2K17)
MATHEMATICS
M304T : Linear Algebra

Time : 3 Hours Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Define an algebra of linear transformation. If A is an algebra with unit element over a field F, then show that A is isomorphic to a sub-algebra of A(V) for some vector space V over F.

   b) Give an example to show that $ST \neq TS$ for $S, T \in A(V)$.

   c) Define a regular and singular transformation. If V is finite dimensional vector space over a field F, then prove that any linear transformation $T \in A(V)$ is singular if and only if there exist a vector $v \in V$ with $v \neq 0$ such that $T(v) = 0$.

   (6+4+4)

2. a) Define the rank of $T \in A(V)$. If V is a finite dimensional vector space over a field F and $S, T \in A(V)$, then prove that

   i) $r(ST) \leq r(T)$

   ii) $r(ST) \leq \min \{r(T), r(S)\}$

   iii) $r(ST) = r(TS) = r(T)$, for S is regular in A(V).

   b) If V is a finite dimensional vector space over F, then prove that $T \in A(V)$ is regular if and only if T maps V onto itself.

   c) If $\lambda \in F$ is a characteristic root of $T \in A(V)$. Prove for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$. **(6+4+4)**

3. a) Define the composition of linear transformation. Show that the product of two linear transformations is a linear transformation.

   b) Define the change of co-ordinate matrix. Let $\mathscr{B} = \{b_1, b_2\}$, $\mathcal{C} = \{c_1, c_2\}$ be two basis with $b_1 = 4c_1 + c_2$, $b_2 = -6c_1 + c_2$. Suppose $x = 3b_1 + b_2$. Then find $[x]_{\mathcal{C}}$.

   c) Define a linear functional and dual basis with examples. Let V be a vector space over a field F. Then prove that the double dual V^{**} is isomorphic to V.

   (4+4+6)

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4. a) Define an invariant subspace of a vector space V. If $W \subseteq V$ is invariant under T, then prove that T induces a linear transformation $\overline{T}$ on $\overline{V}$. If T satisfies a polynomial $q(x) \in F[x]$, then prove that $\overline{T}$ also satisfies q(x). Further, if $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and p(x) is that for T, then prove that $p_1(x)|p(x)$.

   b) Define the canonical linear transformation. If $T \in A(V)$ has all its characteristic roots in F, then prove that there exist a basis of V in which the matrix of T is triangular. **(7+7)**

5. a) Define a nilpotent linear transformation. If $T \in A(V)$ is nilpotent, then show that $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_m T^m$ is invertible, when $\alpha_0 \neq 0$, where $\alpha_i \in F$.

   b) If $T \in A(V)$ is a nilpotent linear transformation, then show that the invariants of T are unique.

   c) Let $T \in A(V)$ has a minimal polynomial $p(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{r-1} x^{r-1} + x^r$ over F. Suppose that V is a module in a cycle module relative to T, then prove that there exist a basis of V over F such that the matrix of T in this basis is of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{r-1} \end{pmatrix}$$

(4+4+6)

6. a) Let u and v be two vectors in an inner product space V such that $||u + v|| = ||u|| + ||v||$. Prove that u and v are linear dependent vectors. Give an example to show that the converse of this statement is not true.

   b) State and prove the Gram-Schmidt orthogonalization process.

   c) Define an orthogonal complement. Let $u = (-1, 4, -3)$ be a vector in the inner product space with standard inner product. Find a basis of the subspace $u^{\perp}$ of R^3. **(4+6+4)**

7. a) Define the quadratic form. Explain the classification of quadratic forms with suitable examples.

b) Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of quadratic form subject to $x^T x = 1$ and a unit vector at which this value is attained.

c) Explain the singular value decomposition. **(6+4+4)**

8. a) Define symmetric bilinear form with an example. Let $\mathbb{B}$ be a bilinear form on a finite dimensional vector space V and let $\mathscr{B}$ be an ordered basis of V. Then show that $\mathbb{B}$ is symmetric if and only if $\psi_{\mathscr{B}}(\mathbb{B})$ is symmetric.

b) Show that two real symmetric matrices are congruent if and only if they have same rank and signature. **(7+7)**

Third Semester M.Sc. Degree Examination, May 2024
(CBCS – Y2K17)
MATHEMATICS
M303T : Functional Analysis

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five full** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define a Banach space. State and prove Holder's and Minkowski's inequalities for l_p^n space $(1 \le p < \infty)$.

   b) Let M be a closed linear subspace of a normed linear space N. Show that the quotient space N/M is also normed linear space with the norm of each coset x + M defined as $\|x + M\| = \inf\{\|x + m\| : m \in M\}$. Further if N is a Banach space, then prove that N|M is also a Banach space. **(9+5)**

2. a) Let $(B, \|.\|)$ be a Banach space such that $B = M \oplus N$, where M and N are linear subspaces of B. Then show that $B_1 = (B, \|.\|_1)$ is a Banach space if M and N are closed in B, where $\|z\|_1 = \|x + y\|_1 = \|x\| + \|y\|, \forall z \in B$.

   b) Show that the set B(N, N′) of all continuous linear transformations from a normed linear space N into N′ is itself a normed linear space. Further show that B(N, N′) is complete when N′ is complete. **(6+8)**

3. a) State open mapping theorem and prove closed graph theorem.

   b) If M is a closed linear subspace of a normed linear space N and $x_0 \notin M$, $x_0 \in N$ then prove that there exist $f_0 \in N^*$ such that $f_0(m) = 0$, $f_0(x_0) \neq 0$. **(8+6)**

4. a) Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non-empty set of continuous linear transformations from B to N with the property that $\{T_i(x)\}$ is a bounded subset of N for each x in B, then prove that $\{\|T_i\|\}$ is a bounded set of numbers.

   b) Show that the mapping $T \to T^*$ is an isometric isomorphism of B(N) into $B(N^*)$ which reverses the product and preserves the identity transformation, where T is an operator on N and T^* is an operator on N^*.

   c) Let N be a normed linear space and x be a non-empty subset of N. If f(x) is bounded for each $f \in N^*$ then show that x is bounded. **(6+5+3)**

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5. a) Define orthogonal complement of set $S \subset H$. Prove the following.
   i) $\{0\}^{\perp} = H$ and $H^{\perp} = \{0\}$
   ii) $S \cap S^{\perp} \subseteq \{0\}$
   iii) $S_1 \subseteq S_2 \Rightarrow S_1^{\perp} \supseteq S_2^{\perp}$
   iv) $S^{\perp}$ is a closed linear subspace of H
   v) $S \subseteq (S^{\perp})^{\perp} = S^{\perp\perp}$.

   b) Define translation. Prove that translation preserves convexity and closure. **(10+4)**

6. a) If S is a non-empty subset of a Hilbert space H, then prove that $S^{\perp}$ is a closed linear subspace of H.

   b) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

   c) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed. **(5+5+4)**

7. a) Define the adjoint of an operator T on H and prove the following.
   i) $(T_1 + T_2)^* = T_1^* + T_2^*$
   ii) $(\alpha T)^* = \overline{\alpha}\, T^*$
   iii) $\|T^*\| = \|T\|$
   iv) T is non-singular implies T^* is non-singular.

   b) Show that eigen vectors associated with distinct eigen values of a self adjoint operator are orthogonal.

   c) Let H be an inner product space and $S = \{e_1, e_2, ..., e_n\}$ an orthonormal subset of H. If $x \in$ span (s), then show that $x = \sum_{i=1}^{n} < x, e_i > e_i$. **(6+4+4)**

8. a) If $P_1, P_2, ..., P_n$ are projections on closed linear subspaces $M_1, M_2, ..., M_n$ of a Hilbert space H, then prove that $P_1 + P_2 + ... + P_n$ is a projection iff P_i's are pair wise orthogonal.

   b) State and prove spectral theorem. **(7+7)**

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III Semester M.Sc. Degree Examination, May 2024
(CBCS – Y2K17)
MATHEMATICS
M302T : Fluid Mechanics

Time : 3 Hours Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) Define Kronecker δ-symbol. Prove that $\delta_{ij} = \alpha_{ip}\,\alpha_{jp}$, where α_{ij} are the components of the matrix of the transformation from x_i system to the x_i' system.

b) Let $\vec{a}$ and $\vec{b}$ be two vectors with components a_i and b_i respectively. Let A be a tensor with components a_{ij}. Then show that $a_i b_i$ and a_{ii} are scalar invariants.

c) Let $\vec{a}$ and $\vec{b}$ be two vectors, A be a tensor and I be an identity tensor. Then prove the following.

i) $I\vec{a} = \vec{a}$

ii) $IA = AI = A$

iii) $I.A = A.I = \text{trace}(A)$

iv) $I.\vec{a}\otimes\vec{b} = \vec{a}\otimes\vec{b}.I = \text{trace}(\vec{a}\otimes\vec{b}) = \vec{a}.\vec{b}$. **(5+5+4)**

2. a) If $\vec{w}$ is the dual vector of a skew tensor A, then show that $|\vec{w}| = \frac{1}{\sqrt{2}}|A|$.

b) Find the spherical and deviatoric parts of the tensor $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

c) If $f_k = f_k(x_i)$ then prove the following.

i) $(x_k f_k)_{,i} = f_i + x_k f_{k,i}$

ii) $(x_k f_k)_{,ij} = f_{i,j} + f_{j,i} + x_k f_{k,ij}$

iii) $\nabla^2 (x_k f_k) = 2f_{k,k} + x_k (\nabla^2 f_k)$. **(4+4+6)**

P.T.O.

3. a) Write a note on Directional Derivative and normal derivative.

   b) Let $\vec{u} = \alpha\left(x_1^2 x_2 e_1 + x_2^2 x_3 e_2 + x_3^2 x_1 e_3\right)$, where α is a constant. Then find

      i) the gradient of $\vec{u}$.

      ii) the divergence of $\nabla\vec{u}$ and $\nabla\vec{u}^T$.

      iii) the curl of $\nabla\vec{u}^T$.

   c) Define :

      i) Lagrangian description of motion.

      ii) Eulerian description of motion. **(4+6+4)**

4. a) Deduce the formula for the material derivative of circulation.

   b) Let $\vec{s}\left(\vec{n}\right)$ be the stress vector, T represents Cauchy's stress tensor and $\vec{n}$ is exterior normal vector to the slant face of the tetrahedron. Then show that $\vec{s}\left(\vec{n}\right) = T^T\vec{n}$. **(5+9)**

5. a) Obtain the expression for the law of conservation of angular momentum in its standard form.

   b) Establish the Eulers equation of motion in its most general form. **(8+6)**

6. a) Derive temperature equation for an incompressible viscous fluid.

   b) Obtain the exact solution of the Navier Stokes equation for the plane-poiseuille flow. Further explain mathematically

      i) Maximum velocity

      ii) Average velocity

      iii) Shearing stress on the wall. **(6+8)**

7. Formulate the Stoke's second problem and solve it. **14**

8. a) In two-dimensional flow field is given by $\varphi = xy$. Then

      i) Show that the flow is irrotational.

      ii) Find the velocity potential and verify that ϕ and ψ satisfy the Laplace equation.

      iii) Find stream line and potential line.

   b) State and prove Blasius theorem. **(7+7)**

PG – 443

III Semester M.Sc. Degree Examination, May 2024
(CBCS – Y2K17 Scheme)
MATHEMATICS
M301 T : Differential Geometry

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let $V = y^2U_1 - xU_3$ and let $f = xy$, $g = z^3$. Then compute

i) $V[fg]$

ii) $V[g] - g\,V[f]$

iii) $V[V[f]]$.

b) If β is the reparametrisation of α by h, then prove $\beta'(s) = \frac{dh}{ds}(s)\alpha'(h(s))$.

c) Reparametrize the curve $\alpha(t) = (e^t, e^{-t}, \sqrt{2}\,t)$ by $h(s) = \log s$ on $J : s > 0$.

(6+4+4)

2. a) Evaluate the 1-form $\varphi = x^2\,dx - y^2\,dz$ on the vector fields.

i) $V = xU_1 + yU_2 + zU_3$

ii) $W = xy\,(U_1 - U_3) + yz\,(U_1 - U_2)$

iii) $\frac{1}{x}V + \frac{1}{y}W$.

b) Verify the Leibniz formula $d(\varphi \wedge \psi) = d\varphi \wedge \psi - \varphi \wedge d\psi$ for $\varphi = \frac{dx}{y}$ and $\psi = z\,dy$.

c) Find $F_*(V_P)$ for the mapping $F = (x\cos y, x\sin y, z)$ and $V = (2, -1, 3)$, $P = \left(2, \frac{\pi}{2}, \pi\right)$.

(6+4+4)

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3. a) Compute the Frenet apparatus κ, τ, T, N, B of the unit speed curve $\beta(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$. Show that this curve is a circle and find its center and radius.

b) Let $V = -yU_1 + xU_3$ and $W = \cos x\, U_1 + \sin x\, U_2$. Then compute $\nabla_V W$, $\nabla_V(Z^2 W)$, $\nabla_V(\nabla_V W)$, $\nabla_V V$, $\nabla_W V$, $\nabla_V(xV - ZW)$. **(6+8)**

4. a) Compute connection forms for cylindrical frame field.

b) If S and T are translations, then prove the following :

i) ST = TS is a translation.

ii) for p, q $\in E^3$, there exist unique translation T such that T(p) = q.

c) If F is an isometry of E^3, then prove that there exist a unique translation T and a unique orthogonal transformation C such that F = TC. **(4+4+6)**

5. a) Show that $X : E^3 \to E^3$; $X(u, v) = (u + v, u - v, uv)$ is a proper patch and that the image of X is the surface $M : z = \frac{x^2 - y^2}{4}$.

b) Show that a mapping $X : D \to E^3$ is regular if and only if the u, v – partial derivatives $X_u(d)$, $X_v(d)$ are linearly independent for all $d \in D$, where D is an open set in E^2.

c) Find a parametrization for the entire surface obtained by revolving the curve

i) C : y = coshx around x-axis

ii) $C : (z - 2)^2 + y^2 = 1$ around the y-axis. **(4+6+4)**

6. a) If $M : g(x, y, z) = c$ is a surface in E^3, then show that the gradient vector field $\nabla g = \sum\left(\frac{\partial g}{\partial x_i}\right)U_i$ is a non-vanishing normal vector field on the entire surface M.

b) Let η be a 2-form on a surface M and let V and W be linearly independent tangent vectors at some point $p \in M$. Then prove that $\eta(aV + bW, cV + dW) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\eta(V, W)$.

c) Let $X : D \to M$ be a co-ordinate patch in a surface M. For any 1-form φ on M, 2-form γ on M, show that

i) $X^*(\varphi) = \varphi(X_u)du + \varphi(X_v)dv$

ii) $X^*(\gamma) = \gamma(X_u, X_v)\, du\, dv$. **(5+3+6)**

7. a) Obtain shape operator of :

i) a sphere

ii) a cylinder

iii) a plane.

b) If p is a non-umbilic point ($k_1 \neq k_2$), then prove that there are exactly two principal directions and these are orthogonal. Further if e_1 and e_2 are principal vectors in these directions; then $S(e_1) = k_1 e_1$, $S(e_2) = k_2 e_2$. **(6+8)**

8. a) If X is a patch in $M \subset E^3$, then prove that $l = S(X_u).X_u = U.X_{uu}$, $m = S(X_u).X_v = U.X_{uv}$, $n = S(X_v).X_v = U.X_{vv}$.

b) Compute K and H and hence k_1, k_2 for the surface $X(u, v) = (u \cos v, u \sin v, bv)$, $b \neq 0$.

c) Find the curvature of the ellipsoid $M : g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **(4+5+5)**
